S

93202Q

NEW ZEALAND QUALIFICATIONS AUTHORITY
MANA TOHU MĀTAURANGA O AOTEAROA

Scholarship 2005
Mathematics with Calculus

9.30 am Saturday 10 December 2005
Time allowed: Three hours
Total marks: 120

QUESTION BOOKLET

A 4-page booklet (S-CALCF) containing mathematical formulae and tables has been centre-stapled in the middle of this booklet. Before commencing, carefully detach the Formulae and Tables Booklet and check that none of its pages is blank.

Answer ALL questions.

Write ALL your answers in the Answer Booklet 93202A.

Check that this Question Booklet 93202Q has pages 2–7 in the correct order and that none of these pages is blank.

YOU MAY KEEP THIS BOOKLET AT THE END OF THE EXAMINATION.

You should attempt ALL six questions. Show ALL your working.

QUESTION ONE (20 marks)

(a) If α is a complex root of the equation $z^5 = 1$, show that $\alpha + \alpha^2 + \alpha^3 + \alpha^4 = -1$.
Hence find the equation with real coefficients whose roots are $\alpha + \alpha^4$ and $\alpha^2 + \alpha^3$.

(6 marks)

(b) The roots z_i $(i = 1, 2, 3, 4, 5)$ of $z^5 - 32 = 0$ are represented in an Argand diagram by the points A, B, C, D, and E, respectively, taken anticlockwise, with $|\arg z_2| = |\arg z_5|$ and $|\arg z_3| = |\arg z_4|$, and A = (2,0).

(i) Sketch an Argand diagram showing clearly the points A, B, C, D, and E.

Find the distance between the points represented by C and E in terms of b, the y-coordinate of B.

(6 marks)

(ii) A variable complex number w is given by $w = t(1 + i)$, $t \in R$, and w is represented in the Argand diagram by the variable point W.

If F is the point (1,–1), find the exact value of the minimum distance AW + FW.
(You do not need to prove that it is a minimum.)

(8 marks)

QUESTION TWO (12 marks)

(a) A circle, with centre (a,b) and radius r, is 'squashed' in the y-direction by a scale factor of h, to form an ellipse (as shown in **Fig. 1**). This transforms a point A(x,y) on the circle to the point B where

$$\text{BC} = h\text{AC}, \qquad 0 < h < 1.$$

For $h = \frac{1}{2}$, show that when one of the points of intersection of the circle and the ellipse lies on the y-axis, then $b^2 = 9(r^2 - a^2)$.

(6 marks)

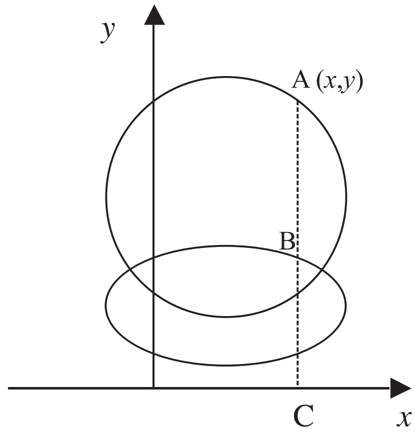

Fig. 1

(b) **Fig. 2** shows the circle $x^2 + y^2 = r^2$ and the ellipse $x^2 + 16(y-r)^2 = r^2$. Find, in terms of r, the area between the ellipse and the circle (shaded in **Fig. 2**).

[You may use the substitution $x = r\sin u$ to find the integral $\int \sqrt{r^2 - x^2}\ \mathrm{d}x$.]

(6 marks)

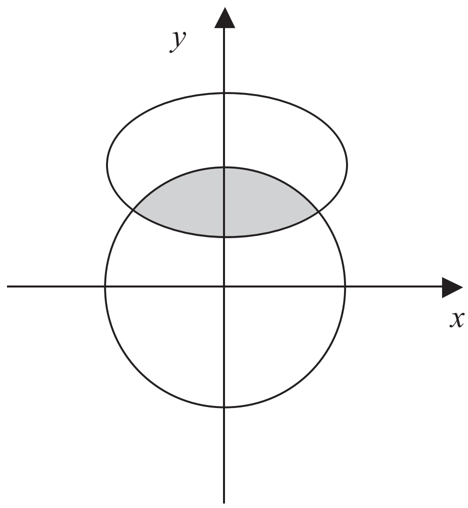

Fig. 2

QUESTION THREE (22 marks)

(a) The shape of a rotor blade for a chemical mixing machine may be modelled by the function

$$y = k + \frac{1}{m}\ln\left(\frac{k}{x}\right), \quad \text{with } 0 < y \le a,$$

where m and k are positive constants (see **Fig. 3**).

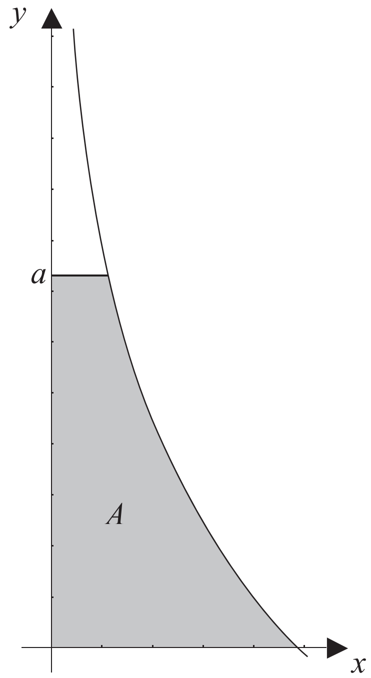

Fig. 3

(i) Find the value of a, in terms of m and p, when the blade's area, A, is p% of the maximum possible area, $\frac{k}{m}e^{mk}$.

(6 marks)

(ii) When the blade rotates in the chemical, it traces out a path equivalent to a rotation of the graph of the function $y = k + \frac{1}{m}\ln\left(\frac{k}{x}\right)$ through 2π about the y-axis. If V is the volume of revolution, then it is necessary for the stability of the blade that

$$\frac{V}{A} \le e^{mk}.$$

Show that in this case the minimum value of a is $\frac{1}{m}\ln\left(\frac{\pi k}{2 - \pi k}\right)$.

(8 marks)

(b) Find $\frac{dy}{dx}$ when $y = x\sin nx + \frac{1}{n}\cos nx$, where n is a positive constant.

Hence, or otherwise, evaluate $I_n - I_{n-1}$, where

$$I_n = \int_0^{\pi} \left(\tfrac{1}{2}\pi - x\right)\sin\left(\left(n + \tfrac{1}{2}\right)x\right)\operatorname{cosec}\left(\tfrac{1}{2}x\right)dx \quad n \in \{1, 2, 3, ...\}.$$

Hence find I_n in terms of n, given that $I_0 = 0$.

(8 marks)

QUESTION FOUR (20 marks)

(a) If $x = f(t)$, $y = g(t)$ and $\dfrac{d^2y}{dx^2} = 0$,

(i) Prove that $\dfrac{dx}{dt} \cdot \dfrac{d^2y}{dt^2} = \dfrac{dy}{dt} \cdot \dfrac{d^2x}{dt^2}$.

(6 marks)

(ii) Hence, or otherwise, show that the points of inflection of the curve defined parametrically by

$$x = a\cos t + \tfrac{1}{2}b\cos 2t$$
$$y = a\sin t + \tfrac{1}{2}b\sin 2t$$

are given by $\cos t = -\dfrac{a^2 + 2b^2}{3ab}$, where a and b are constants.

(You may assume that $\dfrac{d^3y}{dx^3} \neq 0$).

(6 marks)

(b) For a function $y = f(x)$ find the value of the gradient $\dfrac{dy}{dx}$ at the point where $y = 2$, given that:

$$\frac{dx}{dy} \cdot \frac{d^2y}{dx^2} = k\frac{dy}{dx}$$

where k is a constant and the gradient, $\dfrac{dy}{dx}$, is 1 at the point (0,1).

(8 marks)

QUESTION FIVE (16 marks)

(a) Expand $\cos(2A+B)$ and hence prove that

$$\tfrac{1}{4}\cos 3\theta = \cos^3\theta - \tfrac{3}{4}\cos\theta.$$

Hence, by putting $x = \tfrac{2}{3}\cos\theta$, find the exact roots of the equation

$$27x^3 - 9x = 1$$

in terms of π.

Hence, without using your calculator, and clearly showing your method, write down the exact value of the product

$$\cos\frac{\pi}{9}\cos\frac{3\pi}{9}\cos\frac{5\pi}{9}\cos\frac{7\pi}{9}.$$

(8 marks)

(b) The function $f(n)$ is defined for $n \in \{1, 2, 3, ...\}$ by:

$$f(n) = \sum_{k=1}^{k=n} \frac{2k-1}{k(k+1)(k+2)} = \frac{1}{1\times2\times3} + \frac{3}{2\times3\times4} + \frac{5}{3\times4\times5} + ... + \frac{2n-1}{n(n+1)(n+2)}.$$

Show that $f(n)$ may also be written in the form

$$f(n) = \sum_{k=1}^{k=n}\left(-\frac{1}{2k} + \frac{3}{k+1} - \frac{5}{2(k+2)}\right)$$

and hence find $f(n)$ in terms of n, and show that $\lim_{n\to\infty} f(n) = \tfrac{3}{4}$.

(8 marks)

QUESTION SIX (30 marks)

(a) Show that the equation $4x^2 - y^2 - 16hx + 2hy + 15h^2 - 4a^2 = 0$ where h and a are positive constants, represents a hyperbola.

If the tangent to this hyperbola at the point (p,q) is parallel to the straight line $y = \left(e^2 - 1\right)x$, where e is the eccentricity of the hyperbola, show that $p - q = h$.

(The eccentricity of the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ is given by $e^2 = 1 + \dfrac{b^2}{a^2}$.) (6 marks)

(b) A beam of light aimed at the focus G = $(-ae,0)$ of a hyperbolic reflector whose equation is $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, will be reflected through the focus F = $(ae,0)$. **Fig. 4** shows the case where the light passes through the point A vertically above F.

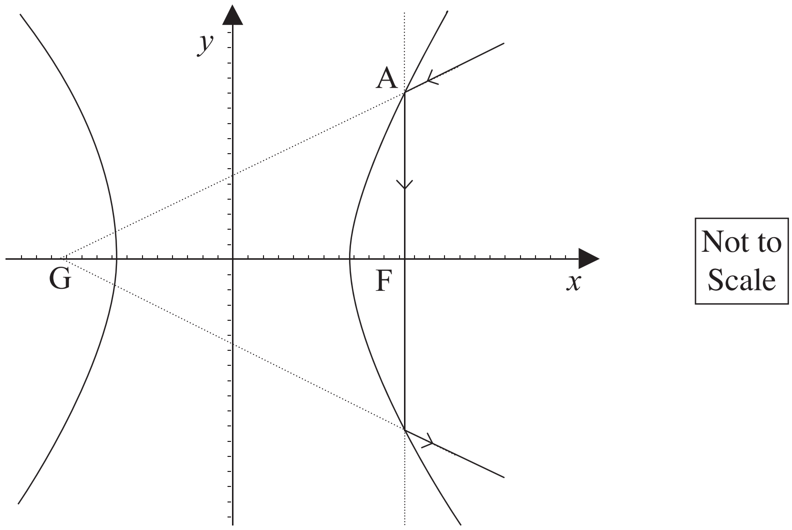

Fig. 4

For the hyperbola given in part (a) above, $4x^2 - y^2 - 16hx + 2hy + 15h^2 - 4a^2 = 0$, a similar beam of light aimed at its focus G_1, is reflected along the corresponding line A_1F_1, perpendicular to the x-axis.
Find the equation of the line A_1G_1 that the light was initially travelling along. (8 marks)

(c) Given that the equation of the tangent at the point (x_1, y_1) on the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{is} \quad \frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1,$$

find the equation of this tangent in terms of a and b and its gradient m. (8 marks)

(d) For the hyperbola $4x^2 - y^2 - 16hx + 2hy + 15h^2 - 4a^2 = 0$, a tangent with positive gradient passes through the point $(\frac{2a}{3} + 2h, h)$.

Show that the size of the angle, α, between this tangent and the asymptote with positive gradient may be expressed as $\alpha = \tan^{-1}\left(\dfrac{2(41 - 15\sqrt{5})}{139}\right)$. (8 marks)